ASYMPTOTIC BEHAVIOUR AND ARTINIAN PROPERTY OF GRADED LOCAL COHOMOLOGY MODULES

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ABSTRACT. In this paper, considering the difference between the finiteness dimension and cohomological dimension for a finitely generated module, we investigate the asymptotic behavior of grades of components of graded local cohomology modules with respect to irrelevant ideal; as long as we study some artinian and tameness property of such modules.

1. INTRODUCTION

Throughout the paper, $R = \bigoplus_{n\geq 0} R_n$ is a graded Noetherian ring where the base ring R_0 is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}_0$ and R is generated, as an R_0-algebra, by finitely many elements of R_1. Moreover, we use $\mathfrak{a}_0$ to denote a proper ideal of R_0 and we set $R_+ = \bigoplus_{n>0} R_n$, the irrelevant ideal of R, $\mathfrak{a} = \mathfrak{a}_0 + R_+$, and $\mathfrak{m} = \mathfrak{m}_0 + R_+$. Also, we use M to denote a non-zero, finitely generated, graded R-module. It is well known (cf [5, §12]) that, for each $i \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of all non-negative integers), the i-th local cohomology module $H^i_{R_+}(M)$ of M with respect to R_+ inherits natural grading. For each $n \in \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of integers), we use the notation $H^i_{R_+}(M)_n$ to denote the n-th graded component of $H^i_{R_+}(M)$. Now, according to [5, 15.1.5], for each $i \geq 0$, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all sufficiently large values of n. The cohomological dimension of M with respect to R_+ is denoted by $cd(R_+, M)$. Thus $cd(R_+, M)$ is the largest non-negative integer i such that $H^i_{R_+}(M)$ is non-zero.

The asymptotic behaviors of $H^i_{R_+}(M)_n$ when $n \to -\infty$ constitute lots of interest (see for example [1], [2], [3], [4] and [7]). As a basic reference in this topic we recommend [1]. In [3], it has been shown that the set $\mathrm{Ass}_{R_0}(H^{f_{R_+}(M)}_{R_+}(M)_n)$ is asymptotically stable, as $n \to -\infty$, where $f_{R_+}(M)$, the finiteness dimension of M with respect to R_+, is the least

Date: 30 July 2008.

2000 *Mathematics Subject Classification.* 13D45, 13A02, 13E10.

Key words and phrases. Graded local cohomology, Asymptotic behavior , Artinian modules.

non-negative integer i such that $H^i_{R_+}(M)$ is not finitely generated. It therefore follows that $\mathrm{grade}(\mathfrak{a}_0, H^{f_{R_+}(M)}_{R_+}(M)_n) < \infty$ for all $n \ll 0$. Recently, M. Brodmann raised the question whether the sequence $(\mathrm{grade}(\mathfrak{a}_0, H^{f_{R_+}(M)}_{R_+}(M)_n))_n$ of integers is stable, as $n \to -\infty$. One of the main proposes of this paper is to provide an affirmative answer for this question in certain cases. In particular, in 2.3, we show that if $f_{R_+}(M) = cd(R_+, M)$, then $\mathrm{grade}(\mathfrak{a}_0, H^{f_{R_+}(M)}_{R_+}(M)_n) = f^{R_+}_{\mathfrak{a}}(M) - f_{R_+}(M)$ for all $n \ll 0$, where $f^{R_+}_{\mathfrak{a}}(M)$, the $\mathfrak{a}$-finiteness dimension of M relative to R_+, is the least non-negative integer i such that R_+ is not contained in the radical of the ideal $(0 :_R H^i_{\mathfrak{a}}(M))$.

The concept of tameness is the most fundamental concept related to the asymptotic behavior of cohomology. A graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is said to be tame or asymptotically gap free (cf [3, 4.1]) if either $T_n \neq 0$ for all $n \ll 0$ or else $T_n = 0$ for all $n \ll 0$. It is known that any graded Artinian R-module is tame [1, 4.2]. Now, the tameness property of $H^i_{\mathfrak{m}_0R}(H^{f_{R_+}(M)}_{R_+}(M))$ provides some results on the stability of the sequence $(\mathrm{depth}_{R_0}(H^{f_{R_+}(M)}_{R_+}(M)_n))_n$, as $n \to \infty$ (cf corollary 2.6). In general, the finiteness of graded local cohomology module has an important role to study the asymptotic behavior of the n-th graded component $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$, as $n \to -\infty$. One of parts in finiteness of graded local cohomology is Artinianess. In the rest of the paper, we study the Artinianess of the modules $H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M))$. Indeed, Theorem 2.4, which improves the results [9, 2.2] and[4, 4.2], shows that if $j \leq g(M)$, then $H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M))$ is Artinian for $i = 0, 1$. Also, Theorem 2.8, which yields [9, 2.3] as a consequence, settle, under the assumption $cd(R_+, M) - f_{R_+}(M) = 1$, a necessary and sufficient condition for the $R-$modules $H^i_{\mathfrak{m}_0R}(H^{f_{R_+}(M)}_{R_+}(M))$ to be Artinian. Finally, 2.10, establishes some results on the Artinianess of the R-modules $H^i_{\mathfrak{m}_0R}(H^{cd(R_+,M)}_{R_+}(M))$ in the case where $dim(R_0) - 3 \leq i \leq dim(R_0)$.

2. THE RESULTS

We keep the notations and hypotheses introduced in the Introduction. In addition, for a finitely generated R_0-module X with $\mathfrak{a}_0X \neq X$, we use the notation $\mathrm{grade}(\mathfrak{a}_0, X)$ to denote the least integer i such that $H^i_{\mathfrak{a}_0}(X) \neq 0$. In particular the above mentioned number is denoted by $\mathrm{depth}(X)$ whenever $\mathfrak{a}_0 = \mathfrak{m}_0$.

Let $f_{R_+}(M) < \infty$. It is well known (see for example [3, 5.6]) that $\mathrm{Ass}_{R_0}(H^{f_{R_+}(M)}_{R_+}(M)_n)$ is asymptotically stable, as $n \to -\infty$. Therefore, since $H^{f_{R_+}(M)}_{R_+}(M)$ is not finitely generated,

it follows that $\mathfrak{a}_0 H^{f_{R_+}(M)}_{R_+}(M)_n \neq H^{f_{R_+}(M)}_{R_+}(M)_n$ for all $n \ll 0$; hence, for all $n \ll 0$, we have grade$(\mathfrak{a}_0, H^{f_{R_+}(M)}_{R_+}(M)_n) < \infty$. Next, we show that if $f_{R_+}(M) = cd(R_+, M)$, then the sequence (grade$(\mathfrak{a}_0, H^{f_{R_+}(M)}_{R_+}(M)_n))_n$ is asymptotically stable, as $n \to -\infty$.

Remark 2.1. Using a very slight modification of the proof of [5, 15.1.5], one can show that, for all $i \in \mathbb{N}_0$, $H^i_{\mathfrak{a}}(M) = 0$ for all $n \gg 0$. In view of this fact and [7, 2.3] one can see that

$$f^{R_+}_{\mathfrak{a}}(M) = sup\{i : H^j_{\mathfrak{a}}(M)_n = 0 \text{ for all } j < i \text{ and all } n \ll 0\}.$$

With the aid of the above remark and the following proposition, one can deduce that

$$f^{R_+}_{\mathfrak{a}}(M) = inf\{i : H^i_{\mathfrak{a}}(M)_n \neq 0 \text{ for all } n \ll 0\}.$$

Proposition 2.2. *Let $f^{R_+}_{\mathfrak{a}}(M) < \infty$. Then $H^{f^{R_+}_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M)_n \neq 0$ for all $n \ll 0$.*

Proof. Before beginning the proof, we provide some facts which are needed in the course of the proof.

Let y be an indeterminate and let $R'_0 = R_0[y]_{\mathfrak{m}_0[y]}$, $R' = R'_0 \bigotimes_{R_0} R$ and $M' = R'_0 \bigotimes_{R_0} M$. Then the natural homomorphism $R_0 \longrightarrow R'_0$ is faithfully flat and, by [5, 15.2.2(iv)], $H^i_{\mathfrak{a}}(M)_n \bigotimes_{R_0} R'_0 \cong H^i_{\mathfrak{a}R'}(M')_n$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. It therefore follows that $f^{R'_+}_{\mathfrak{a}R'}(M') = f^{R_+}_{\mathfrak{a}}(M)$. Hence, without loss of generality, we may assume that $R_0/\mathfrak{m}_0$ is an infinite field. Next we can use the exact sequences

$$H^i_{\mathfrak{a}}(\Gamma_{R_+}(M)) \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/\Gamma_{R_+}(M)) \longrightarrow H^{i+1}_{\mathfrak{a}}(\Gamma_{R_+}(M))$$

to see that , for each $i \in \mathbb{N}_0$, $H^i_{\mathfrak{a}}(M)_n \cong H^i_{\mathfrak{a}}(M/\Gamma_{R_+}(M))_n$ for all but only finitely many $n \in \mathbb{Z}$. Hence, by replacing M with $M/\Gamma_{R_+}(M)$, we may assume, in addition, that there exists a homogeneous element $x \in R_1$ which is a non-zerodevisor on M. Now, the exact sequence $0 \longrightarrow M(-1) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ yields an exact sequence of R_0-modules

$$H^{i-1}_{\mathfrak{a}}(M)_n \longrightarrow H^{i-1}_{\mathfrak{a}}(M/xM)_n \longrightarrow H^i_{\mathfrak{a}}(M)_{n-1} \xrightarrow{x} H^i_{\mathfrak{a}}(M)_n \qquad (*)$$

for all $n \in \mathbb{Z}$ and $i \in \mathbb{N}_0$.Therefore In the case where $i = f^{R_+}_{\mathfrak{a}}(M) - 1$, the exactness of $(*)$ in conjunction with 2.1 implies that, for all $n \ll 0$, $H^{i-1}_{\mathfrak{a}}(M/xM)_n = 0$; and hence, by 2.1, $f^{R_+}_{\mathfrak{a}}(M/xM) \geq f^{R_+}_{\mathfrak{a}}(M) - 1$.

Now, we prove the assertion by induction on $f = f^{R_+}_{\mathfrak{a}}(M)$. Let $x \in R_1$ be a non-zero devisor on M. Note that there exists $n_0 \in \mathbb{Z}$ such that for all $n \leq n_0$, $H^0_{\mathfrak{a}}(M/xM)_n = 0$.

Therefore the exact sequence $(*)$ yields the monomorphism

$$H^1_{\mathfrak{a}}(M)_{n-1} \xrightarrow{x} H^1_{\mathfrak{a}}(M)_n$$

for all $n \leq n_0$.

If $f = 1$,then, by2.1, there exists a descending sequence of integers, say $\{n_i\}_{i\in\mathbb{N}}$, such that $n_i \leq n_0$ and that $H^1_{\mathfrak{a}}(M)_{n_i} \neq 0$ for all positive integers i. Therefore, using the above mentioned monomorphism, one can see that $H^1_{\mathfrak{a}}(M)_n \neq 0$ for all $n \leq n_0$; hence the assertion is true for $f = 1$.

Now, assume inductively that $f \geq 2$ and that the result has been proved for smaller values of f. Let M be such that $f^{R_+}_{\mathfrak{a}}(M) = f$. By the arguments at the beginning of the proof, we have $f^{R_+}_{\mathfrak{a}}(M/xM) \geq f - 1$. We now consider two cases:

Case 1: $f^{R_+}_{\mathfrak{a}}(M/xM) > f - 1$. In this case, in view of 2.1, there exists $n_0 \in \mathbb{Z}$ such that $H^{f-1}_{\mathfrak{a}}(M/xM)_n = 0$ for all $n \leq n_0$. Therefore the exact sequence $(*)$ induces the exact sequence $0 \longrightarrow H^f_{\mathfrak{a}}(M)_{n-1} \xrightarrow{x} H^f_{\mathfrak{a}}(M)_n$ for all $n \leq n_0$. Now, one can use the same arguments as in the case $f = 1$ to show that $H^f_{\mathfrak{a}}(M)_n \neq 0$ for all $n \leq n_0$.

Case 2: $f^{R_+}_{\mathfrak{a}}(M/xM) = f-1$. In this case, by inductive hypothesis, $H^{f-1}_{\mathfrak{a}}(M/xM)_n \neq 0$ for all $n \ll 0$. While, by 2.1, $H^{f-1}_{\mathfrak{a}}(M)_n = 0$ for all $n \ll 0$. Thus, using $(*)$, we deduce that $H^f_{\mathfrak{a}}(M)_n \neq 0$ for all $n \ll 0$. The result now follows by induction. $\square$

As an application of the above proposition, we establish, as a theorem, the following asymptotic behavior of grade.

Theorem 2.3. *Let $f^{R_+}_{\mathfrak{a}}(M) < \infty$ and suppose that $(f =) f_{R_+}(M) = cd(R_+, M)$. Then,* $\operatorname{grade}(\mathfrak{a}_0, H^f_{R_+}(M)_n) = f^{R_+}_{\mathfrak{a}}(M) - f$ *for all $n \ll 0$.*

Proof. Since $H^i_{R_+}(M)$ is finitely generated for all $i < f$, it is easy to see that there exists $n_0 \in \mathbb{Z}$ such that, for all $n < n_0$, $H^i_{R_+}(M)_n = 0$ for all $i < f$. Therefore, in the Grothendieck's spectral sequence [8, 11.38]

$$(E^{p,q}_2)_n = H^p_{\mathfrak{a}_0R}(H^q_{R_+}(M))_n \overset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M)_n$$

we have, for all $p \in \mathbb{N}_0$, $(E^{p,q}_2)_n = 0$ for all $n < n_0$ and all $q \neq f$. This in conjunction with [5, 13.1.10] implies that $H^p_{\mathfrak{a}_0}(H^f_{R_+}(M)_n) \cong H^{p+f}_{\mathfrak{a}}(M)_n$ for all $n < n_0$ and all $p \in \mathbb{N}_0$. Hence, we may use 2.1 and 2.2 to see that, for all $n \ll 0$, $H^i_{\mathfrak{a}_0}(H^f_{R_+}(M)_n) = 0$ whenever $i < f^{R_+}_{\mathfrak{a}}(M)-f$ and that $H^i_{\mathfrak{a}_0}(H^f_{R_+}(M)_n) \neq 0$ if $i = f^{R_+}_{\mathfrak{a}}(M) - f$. Thus $\operatorname{grade}(\mathfrak{a}_0, H^f_{R_+}(M)_n) = f^{R_+}_{\mathfrak{a}}(M) - f$ for all $n \ll 0$. $\square$

The asymptotic behavior of grade$(\mathfrak{a}_0, H^j_{R_+}(M)_n)$, when $n \to -\infty$ is related to the tameness property of the graded R-modules $H^i_{\mathfrak{a}_0 R}(H^j_{R_+}(M))$. It is known that any graded Artinian R-module is tame ([1; 4.2]). In the rest of the paper we look for the cases in which the R-modules $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$ are Artinian.

Next, we will use the concept of *cohomological finite length dimension of M* with respect to R_+. The concept is defined as [4, 3.1]

$$g(M) := sup\{i \ : \ \forall j < i, \ l_{R_0}(H^j_{R_+}(M)_n) < \infty \ \forall n \ll 0 \ \}.$$

The following theorem may be viewed as an generalization of [4, 4.2] and [9, 2.2], where the cases in which $i = 0$ and $j = 1$ are justified respectively.

Theorem 2.4. *Let $g(M) < \infty$. Then $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$ is an Artinian R-module for $i = 0, 1$ and $j \leq g(M)$.*

Proof. In view of Kirby's Artinian criterion for graded modules [6, Theorem1], we have to show that, for $i = 0, 1$, the following statements hold.
$(i) H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n$ is an Artinian R_0-module for all $n \in \mathbb{Z}$.
$(ii) H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n = 0$ for all $n \gg 0$.
$(iii) 0 :_{H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n} R_1 = 0$ for all $n \ll 0$.
Since, by [5, 13.1.10] , $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n \cong H^i_{\mathfrak{m}_0}(H^j_{R_+}(M)_n)$ for all $i \geq 0$ and all $n \in \mathbb{Z}$, we may use [5, 7.1.3] and [5, 15.1.5] to see that (i) and (ii) hold. So, we only need to prove (iii). To this end, consider the Grothendieck's spectral sequence [8, 11.38]

$$(E_2^{p,q})_n = H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M))_n \overset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M)_n.$$

Using the concept of $g(M)$, it is easy to see that there exists $n_0 \in \mathbb{Z}$ such that, for all $n < n_0$, $(E_2^{p,q})_n = 0$ for all $q < g(M)$ and all $p \in \mathbb{N}$. Now, the convergence of the above spectral sequence implies that $H^0_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n \cong H^j_{\mathfrak{m}}(M)_n$ for all $n < n_0$. Therefore, in view of [5, 7.1.3], (iii) holds for $i = 0$. While, in the case where $i = 1$, the above spectral sequence yields a monomorphism

$$H^1_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n \longrightarrow H^{j+1}_{\mathfrak{m}}(M)_n$$

for all $n < n_0$. Again, using the fact that $H^{j+1}_{\mathfrak{m}}(M)$ is an Artinian graded R-module, we have $0 :_{H^{j+1}_{\mathfrak{m}}(M)_n} R_1 = 0$ for all $n \ll 0$; so that, $0 :_{H^1_{\mathfrak{m}_0 R}(H^j_{R_+}(M))_n} R_1 = 0$ for all $n \ll 0$, which completes the proof. $\square$

Remark 2.5. The Artinianness of $H^0_{\mathfrak{m}_0R}(H^{g(M)}_{R_+}(M))$ has already been studied in [4]. Also, in [9], Sazeedeh shows that $H^1_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian. As we mentioned above, these results are consequences of theorem 2.4. Moreover, the following example, which has already been presented in [9, 2.9], shows that 2.4 is no longer true for $i = 2$.

Example. Let k be a field and x, y, t be indeterminates. Let $R_0 = k[x, y]_{(x,y)}$, $\mathfrak{m}_0 = (x, y)R_0$ and $R = R_0[\mathfrak{m}_0t]$, the Ress ring of $\mathfrak{m}_0$. Then, $H^2_{\mathfrak{m}_0R}(H^1_{R_+}(R))$ is not Artinian.

As an application of 2.4 we have the following corollary.

Corollary 2.6. *Let $g(M) < \infty$ and let $j \leq g(M)$. Then one of the following statements hold:*
(i) $\operatorname{depth}_{R_0}(H^j_{R_+}(M)_n) = 0$ *for all* $n \ll 0$*;*
(ii) $\operatorname{depth}_{R_0}(H^j_{R_+}(M)_n) = 1$ *for all* $n \ll 0$*;*
(iii) $\operatorname{depth}_{R_0}(H^j_{R_+}(M)_n) \geq 2$ *for all* $n \ll 0$.

Proof. since , by 2.4, $H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M))$ is tame whenever $i = 0, 1$. The result follows immediately. □

Remark 2.7. If $f := f_{R_+}(M) = cd(R_+, M)$, then one can use the same argument as employed in the proof of 2.3 to see that $H^p_{\mathfrak{m}_0}(H^f_{R_+}(M))_n \cong H^{p+f}_{\mathfrak{m}_0+R_+}(M)_n$ for all $n \ll 0$ and all $p \in \mathbb{N}_0$. Hence, using Kirby's Artinian criterion ([6, Theorem1]), we deduce that the R-module $H^p_{\mathfrak{m}_0R}(H^f_{R_+}(M)$ is Artinian. Note that this result was obtained in [9, 2.6] under the extra condition that R_+ is principal.

The next theorem, which is motivated by the above remark, provides a necessary and sufficient condition for R-modules $H^i_{\mathfrak{m}_0R}(H^{f_{R_+}(M)}_{R_+}(M))$ to be Artinian in the case where $cd(R_+, M) - f_{R_+}(M) = 1$. This theorem has been proved in [9, 2.3] under the further condition that the arithmetic rank of R_+ is two.

Theorem 2.8. *Let $i \in \mathbb{N}_0$ and let $cd(R_+, M) = f_{R_+}(M) + 1$. Then $H^i_{\mathfrak{m}_0R}(H^{cd(R_+,M)}_{R_+}(M))$ is an Artinian R-module if and only if $H^{i+2}_{\mathfrak{m}_0R}(H^{f_{R_+}(M)}_{R_+}(M))$ is an Artinian R-module.*

Proof. Set $f = f_{R_+}(M)$ and $c = cd(R_+, M)$. Consider the spectral sequence $(E^{p,q}_2)_n = H^p_{\mathfrak{m}_0R}(H^q_{R_+}(M))_n \overset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M)_n$. As in the proof of 2.3, there exists $n_0 \in \mathbb{Z}$ such that $(E^{p,q}_2)_n = 0$ for all $n < n_0$ and all $q < f$. On the other hand $(E^{p,q}_2)_n = 0$ for all $n \in \mathbb{Z}$ and

all $q > c$. Therefore, the above mentioned spectral sequence induces an exact sequence of R_0-modules and R_0-homomorphisms

$$0 \longrightarrow (E_\infty^{i,c})_n \longrightarrow H^i_{\mathfrak{m}_0R}(H^c_{R_+}(M))_n \xrightarrow{(d_2^{i,c})_n} H^{i+2}_{\mathfrak{m}_0R}(H^f_{R_+}(M))_n \longrightarrow (E_\infty^{i+2,f})_n \longrightarrow 0$$

for all $i \in \mathbb{N}_0$ and all $n < n_0$. To prove the assertion, as in the proof of 2.4, we employ Kirby's Artinian criterion for graded modules. It is clear, by the same argument which is used in the proof of 2.4, that the conditions (i) and (ii) of the criterion are satisfied. So, it is enough for us to verify the condition (iii). Note that, for each $i, j \in \mathbb{N}_0$, $E_\infty^{i,j}$ is a subquotient of the Artinian graded R-module $H^{i+j}_{\mathfrak{m}}(M)$. Therefore if $H^i_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ is Artinian, then, by Kirby's Artinian criterion, there exists $N \in \mathbb{Z}$ such that $0 :_{(E_\infty^{i+2,f})_n} R_1 = 0 = 0 :_{im(d_2^{i,c})_n} R_1$ for all $n < N$. Now, we can use the above exact sequence to see that $0 :_{H^{i+2}_{\mathfrak{m}_0R}(H^f_{R_+}(M))_n} R_1 = 0$ for all $n < min\{n_0, N\}$, which, in turn, in conjunction with Kirby's Artinian criterion implies that $H^{i+2}_{\mathfrak{m}_0R}(H^f_{R_+}(M))$ is Artinian. The proof of the reverse implication is similar to the above proof. □

Remark 2.9. Note that, by the same arguments, 2.7 and 2.8 remains true if we replace $f_{R_+}(M)$ by $g(M)$.

Finally, we get to the edge of the double complex arised from the spectral sequence $(E_2^{p,q})_n = H^p_{\mathfrak{m}_0R}(H^q_{R_+}(M))_n \overset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M)_n$. The next theorem improves some of the already known facts for nearer point [9, 2.8] and give some conditions to investigate the Artinian property for some farther points.

Theorem 2.10. *Set $c = cd(R_+, M)$ and $d = dim(R_0)$. Then:*
$(i) H^d_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ and $H^{d-1}_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ are Artinian R-modules.
(ii) If $H^{d-3}_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ and $H^{d-2}_{\mathfrak{m}_0R}(H^{c-1}_{R_+}(M))$ are Artinian, then $H^d_{\mathfrak{m}_0R}(H^{c-2}_{R_+}(M))$ and $H^{d-1}_{\mathfrak{m}_0R}(H^{c-1}_{R_+}(M))$ are Artinian.
(iii) If $H^{d-3}_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ is Artinian, then $H^{d-1}_{\mathfrak{m}_0R}(H^{c-1}_{R_+}(M))$ is Artinian.
(iv) If $H^d_{\mathfrak{m}_0R}(H^{c-2}_{R_+}(M))$ and $H^{d-1}_{\mathfrak{m}_0R}(H^{c-1}_{R_+}(M))$ are Artinian, then $H^{d-3}_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ is Artinian.
(v) $H^{d-2}_{\mathfrak{m}_0R}(H^c_{R_+}(M))$ is Artinian if and only if $H^d_{\mathfrak{m}_0R}(H^{c-1}_{R_+}(M))$ is Artinian.

Proof. Consider the spectral sequence

$$(E_2^{p,q})_n = H^p_{\mathfrak{m}_0R}(H^q_{R_+}(M))_n \overset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M)_n.$$

(i). Since $d_2^{d-1,c} = d_2^{d,c} = d_2^{d-3,c+1} = d_2^{d-2,c+1} = 0$, we have $E_\infty^{d-1,c} \cong H^{d-1}_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ and $E_\infty^{d,c} \cong H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$. Therefore, since, for each $i, j \in \mathbb{N}_0$, $E_\infty^{i,j}$ is a subquotient of the Artinian graded R-module $H^{i+j}_{\mathfrak{m}}(M)$, the assertion follows immediately.
(ii) *and* (iii). Consider the exact sequence

$$0 \longrightarrow E_3^{d-3,c} \longrightarrow H^{d-3}_{\mathfrak{m}_0 R}(H^c_{R_+}(M)) \stackrel{d_2^{d-3,c}}{\longrightarrow} H^{d-1}_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M)) \longrightarrow E_3^{d-1,c-1} \longrightarrow 0. \qquad (\dagger)$$

Since $E_3^{d-1,c-1} \cong E_\infty^{d-1,c-1}$ is Artinian, (iii) follows immediately from the above exact sequence. Next, as $E_4^{d,c-2} \cong E_\infty^{d,c-2}$ and $E_4^{d-3,c} \cong E_\infty^{d-3,c}$, we have an exact sequence

$$0 \longrightarrow E_\infty^{d-3,c} \longrightarrow E_3^{d-3,c} \stackrel{d_3^{d-3,c}}{\longrightarrow} E_3^{d,c-2} \longrightarrow E_\infty^{d,c-2} \longrightarrow 0.$$

Now, we may use the above two exact sequences, together with our Artinian assumption on $H^{d-3}_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$, to see that $E_3^{d,c-2}$ is Artinian. To complete the proof of (ii), consider the exact sequence

$$H^{d-2}_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M)) \stackrel{d_2^{d-2,c-1}}{\longrightarrow} H^d_{\mathfrak{m}_0 R}(H^{c-2}_{R_+}(M)) \longrightarrow E_3^{d,c-2} \longrightarrow 0$$

and use the Artinian assumption on $H^{d-2}_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M))$ to see that $H^d_{\mathfrak{m}_0 R}(H^{c-2}_{R_+}(M))$ is Artinian.
(iv). Note that $E_3^{d,c-2} \cong \frac{H^d_{\mathfrak{m}_0 R}(H^{c-2}_{R_+}(M))}{im(d_2^{d-2,c-1})}$. This fact, in conjunction with the exactness of $(\dagger)$ and our assumptions on $H^d_{\mathfrak{m}_0 R}(H^{c-2}_{R_+}(M))$ and $H^{d-1}_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M))$, implies that $H^{d-3}_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ is Artinian.
(v). Consider the exact sequence

$$0 \longrightarrow E_\infty^{d-2,c} \longrightarrow H^{d-2}_{\mathfrak{m}_0 R}(H^c_{R_+}(M)) \stackrel{d_2^{d-2,c}}{\longrightarrow} H^d_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M)) \longrightarrow E_\infty^{d,c-1} \longrightarrow 0$$

and note that $E_\infty^{d-2,c}$ and $E_\infty^{d,c-1}$ are Artinian R-modules. So that the Artinianess of $H^{d-2}_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ implies the Artinianess of $H^d_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M))$ and vice versa. $\square$

Acknowledgment. The authors would like to thank Professor M. Brodmann for raising the question which was discussed in Theorem 2.3.

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